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Electrical Engineering Research Laboratory
The University of Texas

Report No. 67

10 April 1953

Characteristics of an Elliptical Electromagnetic Resonant Cavity Operating in the TE_{111} Mode

Prepared Under Office of Naval Research Contract Nonr 375(01)

ELECTRICAL ENGINEERING RESEARCH LABORATORY

THE UNIVERSITY OF TEXAS

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CHARACTERISTICS OF AN ELLIPTICAL ELECTROMAGNETIC RESONANT CAVITY OPERATING IN THE TE_{111} MODE

by

Theodore P. Higgins

Prepared Under Office of Naval Research Contract Nonr 375(01)

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TABLE OF SYMBOLS

SYMBOL	DEFINITION	PAGE FIRST USED
D	Average diameter	3
s	Perimeter of the cylinder cross section	3
e	Eccentricity	3
E	Ellipticity	3
a	Major axis of the ellipse	3
b	Minor axis of the ellipse	3
q	Semi-focal distance of the ellipse	3
TE	Transverse Electric (or H wave) No electric component in the axial direction	2
TM	Transverse Magnetic (or E wave) No magnetic component in the axial direction	2
N	Ratio of the minor to the major axis	5
ξ	Radial elliptic coordinate	5
η	Angular elliptic coordinate	5
ξ_o	The value of ξ which satisfies $Jp_1'(c \cosh \xi) = 0$	5
ds_1	Element of arc length	7
$Sp_n(\eta)$	Angular Mathieu function of order n and even or odd as p is e or o	10
$Jp_n(\xi)$	Radial Mathieu function of order n and even or odd as p is e or o	11
$J_r(\xi)$	Bessel function of the first kind, order r	11
rp'_{11}	The first root of the first order radial Mathieu function and even or odd as p is e or o. The prime and subscript are usually omitted	14
H_a	The a component of the magnetic vector	16
E_a	The a component of the electric vector	16

SYMBOL	DEFINITION	PAGE FIRST USED
c	Parameter used for tabulating the Mathieu coefficients	17
λ	The wavelength	17
B	A complex amplitude constant	17
μ	The magnetic permeability	17
ϵ	The dielectric constant	17
k	The wave number, $(2\pi/\lambda)$	18
k_1	Defined on page 18	18
k_3	π/L	18
$E(e)$	The complete elliptic integral	18
p	Abbreviation for $\pi/2$	18
L	Length of the cavity in the axial direction	18
R	D/L	18
Q	The quality factor	23
ω	The angular frequency, $2\pi f$	23
U_ϵ	The electric energy	23
R_s	The surface resistivity	26
δ	The skin depth	30

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PREFACE

This report was originally submitted as a thesis to the Faculty of the Graduate School of The University of Texas in fulfillment of a requirement for the degree of Master of Science in Electrical Engineering.

Since interest in the cavity resonator was occasioned by and the results of the work are pertinent to the Office of Naval Research Contract Nonr 375(01), it was felt that it should be submitted as a technical report under this contract.

Although it would be desirable for the sake of completeness to extend the calculations to larger eccentricities, the main interest in elliptical cavities lies in the small eccentricity region. In this region fall the eccentricities associated with unavoidable deformations of cylindrical cavities. The extreme tedium of the calculations dictates the use of approximations in the calculations. The particular ones developed in this thesis are sufficiently accurate for eccentricities less than 0.4 and are subject to small errors for eccentricities between 0.4 and 0.5.

ABSTRACT

Formulae are derived for the quality factor and resonant wave length of an elliptical resonant cavity operating in the TE_{111} mode. Calculations are made and curves plotted for their variation with change in eccentricity for values of eccentricity less than 0.5. The necessary integrations are numerical using simplifying assumptions.

For both the even and odd modes, the quality factor increased slightly as the eccentricity was increased from zero to a small value. A further increase in the eccentricity causes the quality factor for the even mode to decrease. The range of eccentricity (0 to 0.5) used was not sufficient to show the anticipated decrease for the odd mode. The eccentricity range considered was limited by the approximations used in the method of evaluation. The approximations were considered as fully justified for eccentricities less than 0.4 and subject to some error for eccentricities between 0.4 and 0.5.

CHAPTER I

INTRODUCTION

The purpose of this thesis is to investigate the effect of small amounts of elliptical deformation on the behavior of certain characteristics of a resonant electromagnetic circular cylindrical cavity. Although the circular cylindrical resonant cavity is a special case of the elliptic cylindrical resonant cavity, the elliptical cavity solution cannot be expressed in terms of the cylindrical functions. The Bessel functions used for the cylindrical case are relatively simple and numerous tabulations are available. The Mathieu functions required for the elliptical case are, however, much more complex and very few tabular values have been published. The resonant wavelength and quality factor in the elliptic cylinder considered as a deformed circular cylinder warrant investigation because a physical cylinder may depart from perfectly circular to an extent determined by the manufacturing tolerance; external forces such as mounting brackets could also cause departure from circular. It seems unlikely that the elliptical cylinder cavity would exhibit such decided superiority over the circular cavity as to justify the considerable additional manufacturing difficulties attendant to its use. The calculations for the mode considered here show no advantages peculiar to the elliptical modes.

Calculations were made in 1946 by Kinzer and Wilson [1] to determine the variation of wavelength in certain modes with the ellipticity of the cylinder. Kinzer and Wilson also derived an expression for the quality factor for one value of eccentricity for the TE_{011} mode; this thesis will consider the TE_{111} mode with several values of eccentricity for both odd and even excitation.

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1 J. P. Kinzer and I. G. Wilson, "Some Results on Cylindrical Resonators," Bell System Technical Journal, vol 26, 1947, p 410

CHAPTER II

ELLIPTICAL COORDINATES AND ELLIPTICAL WAVE FUNCTIONS

(1) The Elliptical Cylinder

The dimensions of the cross section of an elliptical cylinder are shown in Figure 1. The quantities 2a and 2b [2] are the major and minor axes respectively; the focal distance is 2q. The perimeter of the ellipse, s, will be kept constant when the eccentricity is varied, and the parameter used will be the "average diameter," D, which is related to the perimeter by the formula:

(1) $$D = \frac{\text{perimeter}}{\pi} = \frac{s}{\pi}$$

It is evident that for the circular case, D is the diameter of the undistorted circle.

The eccentricity, e, is defined as the ratio of the semi-focal distance, q, to the semi-major axis, a. The eccentricity is not measurable directly and there are two other directly related quantities which are often used instead of the eccentricity as a measure of the departure from a circle. One quantity is the ellipticity, E, defined:

(2) $$E = \frac{\text{difference between major and minor diameters}}{\text{major diameter}} = \frac{a - b}{a}$$

The ellipticity is related to the eccentricity by the formula:

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[2] These and all other symbolic abbreviations are defined on page 11

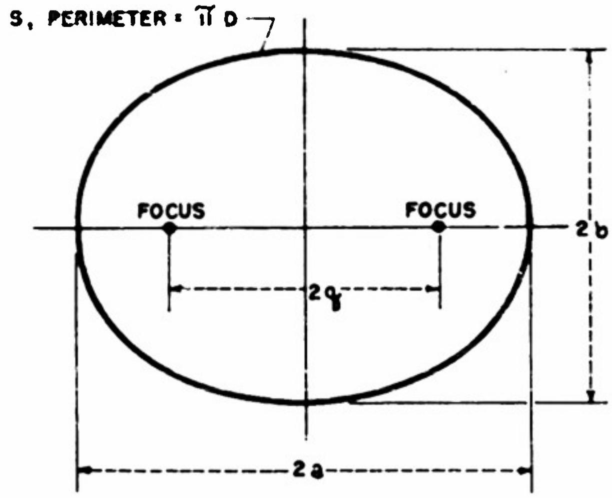

FIGURE 1

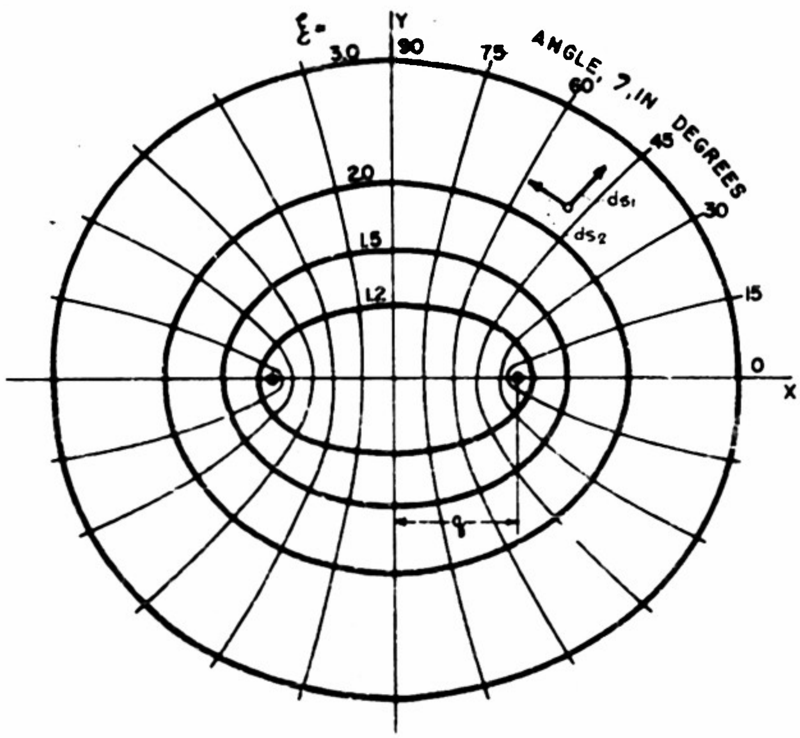

FIGURE 2

(3) $$E = 1 - \sqrt{1 - e^2}$$

The other quantity which may be used to express the departure from circular is N, defined as the ratio of the minor to the major axis.

(4) $$N = b/a = \sqrt{1 - e^2}$$, or, after an elementary hyperbolic trigonometric identity substitution,

$$N = \tanh(\text{arc sech}\, e)$$

The relations (3) and (4) are plotted in Figure 3.

Curves for the variation in wavelength and in the quality factor are plotted against the eccentricity, but values of either E or N can be found by using Figure 3 in conjunction with the curves plotted against the eccentricity.

(2) <u>The Elliptical Coordinate System</u>

The elliptical coordinate system is shown in Figure 2. The orthogonal coordinates ξ and η locate a point uniquely. The elliptical coordinates are related to the x-y coordinates by the transformation equations:

(5) $$x = q \cosh \xi \cos \eta$$
$$y = q \sinh \xi \sin \eta$$

The equation of the boundary surface of the ellipse is:

(6) $$\cosh \xi_0 = \text{constant} = 1/\text{ eccentricity}$$

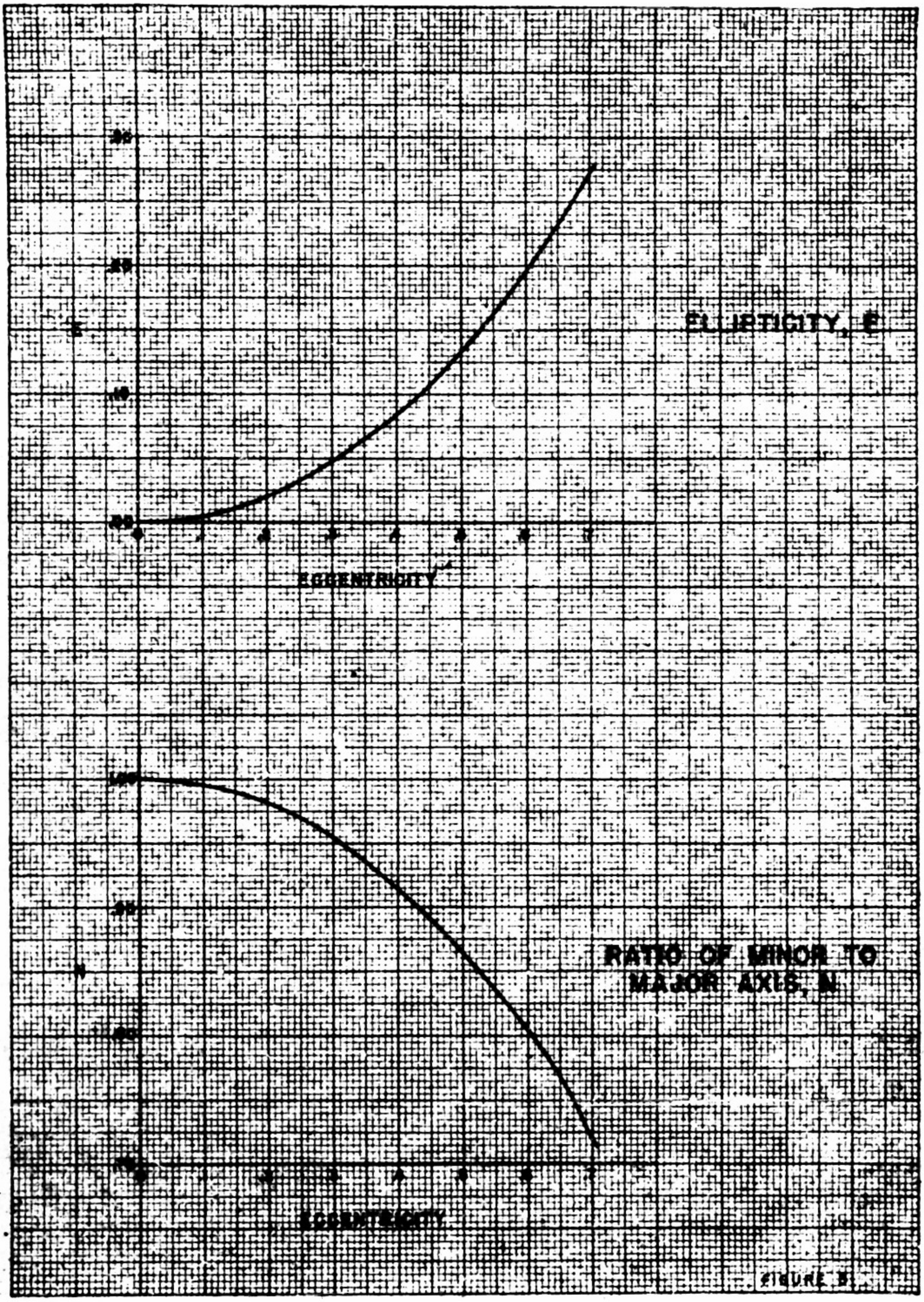
ELLIPTICITY, E
ECCENTRICITY
RATIO OF MINOR TO
MAJOR AXIS, N
ECCENTRICITY

FIGURE 3

The eccentricity, e, varies between zero and one and the ellipse degenerates as $e \to 1$, $\cosh \xi \to 1$, $\xi \to 0$ to a straight line between the focii. The ellipse degenerates into a circle as $e \to 0$, $\cosh \xi \to \infty$, $\xi \to \infty$. The elements of arc length in elliptical cylinder coordinates are ds_1 and ds_2 as shown in Figure 3; the z direction is into the paper with the element of arc length in that direction equal to one dz.

If the definition is made:

(7) $$q_1 = q\,(\cosh^2 \xi - \cos^2 \eta)^{\frac{1}{2}}$$

it will follow that

(8) $$\begin{aligned} ds_1 &= q\,(\cosh^2 \xi - \cos^2 \eta)^{\frac{1}{2}}\, d\xi = q_1\, d\xi \\ ds_2 &= q\,(\cosh^2 \xi - \cos^2 \eta)^{\frac{1}{2}}\, d\eta = q_1\, d\eta \end{aligned}$$

(3) <u>The Wave Equation in Elliptical Coordinates</u>

The two dimensional wave equation in elliptical coordinates is:

(9) $$\frac{\partial^2 f(\xi,\eta)}{\partial \xi^2} - \frac{\partial^2 f(\xi,\eta)}{\partial \eta^2} - k_1^2\,(\cosh^2 \xi - \cos^2 \eta) = 0$$

This equation is known as Mathieu's equation. When a product solution is assumed, and the Bernoulli trial method of separation is followed, the equations separate to give two ordinary differential equations:

(10) $$\frac{\partial^2 f(\eta)}{\partial \eta^2} + (b - k_1^2 q^2 \cos^2 \eta)\ f(\eta) = 0$$

(11) $$\frac{\partial^2 f(\xi)}{\partial \xi^2} - (b - k_1^2 q^2 \cosh^2 \xi)\ f(\xi) = 0$$

where b is the separation constant. The solutions of the first equation, (10), are often called Mathieu functions, and the solutions of the second, (11), are then called associated Mathieu functions. Equation (10) transforms to (11) under the substitution $\eta = \pm i \xi$, and equation (11) transforms to (10) under the substitution $\xi = \pm i \eta$, where i is $\sqrt{-1}$. Solutions exist regardless of the value of the separation constant, b, but the solutions are periodic only for certain characteristic values of the separation constant. Some authors [3] consider only the periodic solutions of Mathieu's equation as Mathieu functions, but more recently, Mathieu functions have been considered as all solutions of (9) whether or not the conditions for periodicity are satisfied. [4] McLachlan [5] has an extensive discussion of solutions where no restrictions are placed on the separation constants. In the calculations which are required in this work, only solutions which are periodic in η will satisfy the required boundary conditions.

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[3] E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, Macmillan, 1948, p 405

[4] Tables Relating to Mathieu Functions, The Computation Laboratory United States National Bureau of Standards, 1951

[5] N. W. McLachlan, Theory and Application of Mathieu Functions, Oxford, 1947

(h) Mathieu Functions

Mathieu functions arise in stability investigations of various mechanical systems, the theory of frequency modulation, loud-speaker theory, and in any electromagnetic or vibration problem which must be stated in elliptical coordinates.

The modern theory of Mathieu functions is credited to Whittaker and much of the subsequent theoretical development is credited to Ince, Strutt, and McLachlan. A historically complete list of 226 references is given in McLachlan.[6] Because both the theory and numerical computations of Mathieu functions are more difficult than those for Bessel and Legendre functions, complete tables which make possible the actual use of Mathieu functions in numerical computation have lagged far behind these other functions and have only recently become available.

The elliptical wave guide was first investigated by Chu[8] in 1938 and following his work, the first numerical tables of coefficients for Mathieu functions was published.[9] Recently the far more accurate and extensive *Tables Relating to Mathieu Functions* have been published.[7] Notation for Mathieu functions

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[6] N. W. McLachlan, *Theory and Application of Mathieu Functions*. Oxford, 1947

[7] *Tables Relating to Mathieu Functions*, The Computation Laboratory, United States National Bureau of Standards, 1951

[8] L. J. Chu, "Electromagnetic Waves in Elliptic Metal Pipes," *Journal of Applied Physics*, vol 9, 1938, p 583

[9] J. A. Stratton, P. M. Morse, L. J. Chu, and R. A. Hutner, *Elliptic Cylinder and Spheroidal Wave Functions*, New York, Wiley, 1941

table is reproduced on page 12. With the exception of the designation of the parameter for the coefficients, all of the notation used here agrees with the notation used in the Tables. Unfortunately it was impossible to use the Tables for numerical calculations because in the mode chosen for investigation, the number of values of the parameter in the desired range was insufficient.

Solutions to equation (10) can be found from the formula:

(12) $$Sp_n(c, \cos\eta) = \sum_{k=0}^{\infty} D_p^n \begin{matrix} \cos k\eta \\ \sin k\eta \end{matrix}$$

where $Sp_n(c, \cos\eta)$ is the angular Mathieu function, p signifies either e (even), or o (odd), n is the order of the function (and for the mode considered always one), c is a parameter which is defined later [10] and $\cos\eta$ is the argument of the function. The coefficients for the right hand summation are found in Stratton [11] or in the Tables. [12] The cosine functions of η are used in the summation for the even functions and sine functions are used in the summation for the odd functions. The series (12) is not a Fourier series because the coefficients are not derived from the Fourier defining integrals, but according to Stratton [13] they apparently satisfy the conditions of convergence necessary for term by term differentiation or integration.

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[10] See page 17
[11] loc. cit. p 78 or p 82. Refer to Table for designation used
[12] loc. cit. tabulated for values of s. Refer to Table for relation between c and s.
[13] loc. cit, p 20

The only angular functions which will be encountered in the TE_{111} mode will be the angular function of the first kind with solutions of the form:

(13) $$Se_1(c,\cos\eta) = \sum_{k=0}^{\infty} De^1_{2k+1} \cos\left[(2k+1)\eta\right] \text{ of period } 2\pi$$

(14) $$So_1(c,\cos\eta) = \sum_{k=0}^{\infty} Do^1_{2k+1} \sin\left[(2k+1)\eta\right] \text{ of period } 2\pi$$

The corresponding radial solutions may be calculated from a joining factor, but in practice, the useful expression which converges much more rapidly than the trigonometric one is expressed as a sum of Bessel functions:

(15) $$Je_1(c,\cosh\xi) = \sqrt{\pi/2} \sum_{k=0}^{\infty} (-1)^k De^1_{2k+1} \mathcal{J}_{2k+1}(c,\cosh\xi)$$

(16) $$Jo_1(c,\cosh\xi) = \sqrt{\pi/2} \tanh\xi \sum_{k=0}^{\infty} (-1)^k (2k) Do^1_{2k+1} \mathcal{J}_{2k+1}(c,\cosh\xi)$$

where $\mathcal{J}_m$ is the Bessel function of the first kind and order m.[14]

[14] Tables Relating to Mathieu Functions, The Computation Laboratory, United States Bureau of Standards, 1951, p xx

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[14] Tables Relating to Mathieu Functions, The Computation Laboratory, United States Bureau of Standards, 1951, p xx

TABLE OF NOTATION AND CONVERSION FACTORS [15]

Notation Used Here	Notation Used in Tables [15]	Notation Used in Stratton [16]	Notation Used in McLachlan [17]	Notation Used in Tang [18]
c^2	s	c^2	$4q$	c^2
$Se_r(c,\cos\eta)$	$Se_r(s,\eta)$	$Se_r^1(c,\cos\eta)$	$ce_r(\eta,q)/A$	$Se_r^1(c,\cos\eta)$
$So_r(c,\cos\eta)$	$So_r(s,\eta)$	$So_r^1(c,\cos\eta)$	$se_r(\eta,q)/B$	$So_r^1(c,\cos\eta)$
De_k^r	De_k^r	D_k^r	A_k^r/A	D_k^r
Do_k^r	Do_k^r	F_k^r	B_k^r/B	F_k^r
$Je_r(c,\cosh\xi)$	$Je_r(s,\xi)$	$Je_r(c,\cosh\xi)$	$Ce_r(\xi,q)/Ag_{er}(s)$	$Re_r(c,\cosh\xi)$
$Jo_r(c,\cosh\xi)$	$Jo_r(s,\xi)$	$Jo_r(c,\cosh\xi)$	$Se_r(\xi,q)/Bg_{or}(s)$	$Ro_r(c,\cosh\xi)$
be_r	be_r	b_r	$a_r + 2q$	b_r
bo_r	bo_r	b_r'	$b_r + 2q$	b_r'

15 loc. cit. p xxxviii except for last column
16 loc. cit.
17 loc. cit.
18 C. C. Tang, "Propagation of Electromagnetic Waves in Hollow Metal Pipes of Elliptical Cross-Section," 1949, University of Texas Thesis

CHAPTER III

A SUMMARY OF OTHER WORK ON ELLIPTIC GUIDES AND CAVITIES

(1) The Results of Chu and Tang

The problem of the propagation of electromagnetic waves in hollow pipes of elliptic cross section has been investigated theoretically by Chu.[19] Chu studied the six lowest order waves. With the exception of modes in the cylindrical pipe which exhibit circular symmetry (TM_{01} and TE_{01}), when the cylinder is deformed to an ellipse both even and odd elliptical modes are generated with their relative magnitude depending on the polarization of the excitation. Because of this splitting, slight deformation of the cylindrical guide may, unlike deformation of the rectangular guide, lead to instability. The cavity considered in this thesis may be regarded as a very short wave guide shorted at the ends so that the generation of two modes does not lead to instability, but, rather, to a broadening of the frequency response of the cavity due to the splitting.

Chu's article covered the theory of elliptical wave guides but omitted much of the numerical calculations which were used in obtaining his results. These numerical calculations were reworked in detail by Tang[20] in 1949. Both Tang and Chu obtained curves for the

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19 L. J. Chu, "Electromagnetic Waves in Hollow Elliptic Pipes of Metal," Journal of Applied Physics, vol 9, September, 1938

20 C. C. Tang, "Propagation of Electromagnetic Waves in Hollow Metal Pipes of Elliptical Cross-Section," 1949, University of Texas Thesis

variation in the cutoff wavelength as a function of the eccentricity for a guide of constant periphery. The curves obtained by Chu are reproduced by Sarbacher and Edson [21] and by Moreno. [22] Both Tang and Chu obtained curves plotted against eccentricity for roots of the equations:

(17)
$$Je\,(r_{e11}) = 0 \quad \text{required for even TM modes}$$
$$Jo\,(r_{o11}) = 0 \quad \text{required for odd TM modes}$$

(18)
$$Je'\,(r'_{e11}) = 0 \quad \text{required for even TE modes}$$
$$Jo'\,(r'_{o11}) = 0 \quad \text{required for odd TE modes}$$

Unfortunately, the accuracy required in the numerical work of Tang and Chu is not sufficient for the calculations of the resonant wavelength and quality factor of a cavity.

(2) Kinzer and Wilson Results on Cylindrical Cavities [23]

Kinzer and Wilson determined the root values of the applicable equations (17) or (18) correct to five significant figures for nine modes in the elliptic cylinder: the even TE_{01n}, the even and odd TM_{11n}, the even and odd TE_{22n}, the even and odd TE_{32n}, the even TM_{01n}, and the even TE_{11n}. The first subscript indicates the number of variations in the angular direction, the second subscript indicates the variations in the radial direction. For a resonant cavity the third subscript indicates the variations in the axial direction, but this does not affect the value of the zeros.

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21 R. L. Sarbacher and W. A. Edson, Hyper and Ultrahigh Frequency Engineering, Wiley, 1943

22 T. Moreno, Microwave Transmission Design Data, McGraw-Hill, 1948

23 J. P. Kinzer and I. G. Wilson, "Some Results on Cylindrical Resonators," Bell System Technical Journal, vol 26, 1947, p 410

Kinzer and Wilson determined an empirical equation for the ratio of the perimeter to the cutoff wavelength for three modes (the even TE_{01n}, the even TM_{11n}, and odd TM_{11n}) as a function of the ellipticity, E.

Kinzer and Wilson derived an expression for Q for one mode and one value of eccentricity (even TE_{01n} with an eccentricity of 0.4814). The circular symmetry of the TE_{01n} makes the calculations necessary to obtain the quality factor simpler than those necessary to find Q for the TE_{111} mode. Their article does not give any details of the methods used to make calculations, but the sparse outline of method of calculations indicates that the procedure was the same as that used in this work, with the exception of a different formula for numerical integration.[24] Since their only result for Q is one point on a curve for a different mode than those considered in this thesis, no numerical comparison can be made of results.

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24 See page 36

CHAPTER IV

DETERMINATION OF THE RESONANT WAVELENGTH

(1) Field equations for the elliptical cylinder cavity

The equations for the components in an elliptical pipe in the TE_{11}[25] mode are:[26]

(19) $$H_z = B\, Sp_1(c, \cos\eta)\, Jp_1(c, \cosh\xi)\, e^{i(\omega t - k_3 z)}$$

$$E_z = 0$$

(20) $$H_\xi = \frac{-k_3}{\omega\mu} E = -B \frac{ik_3}{q_1 k_1^2} Sp_1(\eta)\, Jp_1'(\xi)\, e^{i(\omega t - k_3 z)}$$

(21) $$H_\eta = \frac{-k_3}{\omega\mu} E = -B \frac{ik_3}{q_1 k_1^2} Sp_1'(\eta)\, Jp_1(\xi)\, e^{i(\omega t - k_3 z)}$$

where k_3 is the propagation constant,[27] B is a complex amplitude constant which depends on the relative even and odd modes excited, ω is the angular frequency, μ is the permeability of the dielectric in the guide, and the primes denote either $\frac{\partial}{\partial\xi}$ or $\frac{\partial}{\partial\eta}$.

The boundary conditions require that $E(\xi_0) = 0$ where ξ_0 is the boundary. This implies that

(22) $$Jp'(c, \cosh\xi_0) = 0$$ and with the definition $rp'_{11} = q\sqrt{k_1^2 + \omega^2\mu\epsilon}$

$$\frac{\partial}{\partial\xi} Jp_1(rp'_{11}) = 0$$

---

25 TE (transverse electric) is often written as H (since only non-zero component in the z direction is H_z)

26 J. A. Stratton, Electromagnetic Theory, McGraw-Hill, 1941, p 375

27 k_3, the propagation constant, is often written β

28 In all the following work, the subscripts and primes are omitted from the r designation for root since the only roots which occur will be either r'_{e11} or r'_{o11}

Combining the waves that travel in the positive z direction with those which travel in the negative z direction and making the indicated trigonometric substitutions, the field equations for the TE_{111} mode in the resonant elliptical cylinder are obtained. The time function $e^{i\omega t}$ is suppressed and the equations are:

(23) $$H_z = -B\, k_1^2\, Sp_1(c,\cos\eta)\, Jp_1(c,\cosh\xi)\, \sin(k_3 z)$$

(24) $$H_\xi = \frac{-i\, B\, k_3}{q_1}\, Sp_1(c,\cos\eta)\, Jp_1'(c,\cosh\xi)\, \cos(k_3 z)$$

(25) $$H_\eta = \frac{-i\, B\, k_3}{q_1}\, Sp_1'(c,\cos\eta)\, Jp_1(c,\cosh\xi)\, \cos(k_3 z)$$

(26) $$E_\eta = \frac{B\, k}{q_1}\, Sp_1(c,\cos\eta)\, Jp_1'(c,\cosh\xi)\, \sin(k_3 z)$$

(27) $$E_\xi = \frac{-B\, k}{q_1}\, Sp_1'(c,\cos\eta)\, Jp_1(c,\cosh\xi)\, \sin(k_3 z)$$

(28) $$E_z = 0$$

The radial and angular coefficients are tabulated in Stratton[29] for given values of the parameter, c,[30]

(29) $$c = 2\pi q/\lambda_c$$

---

[29] Stratton, Morse, Chu, and Hutner, Elliptic Cylinder and Spheroidal Wave Functions, Wiley, 1941

[30] The parameter c should not be confused with the c often used to designate the speed of light.

The perimeter of an ellipse, s,[31] is related to the semi-focal distance, q, and the eccentricity, e, by the formula:

(30) $$s = \frac{q}{e}\int_0^{2\pi}\sqrt{1 - e^2\cos^2\eta}\; d\eta$$

(31) $$= \frac{4q}{e}\; E\,(e)^{32}$$

where E (e) is the complete elliptic integral tabulated in Peirce[33] for values of arc sin e.

It will be convenient to use the parameter λ_c/s used by Tang and Chu, and c may be expressed in terms of that parameter by substituting from equation (31) into (29).

(32) $$c = \frac{\frac{\pi}{2}\;\frac{e}{E\,(e)}}{\lambda_c/s}$$

(2) Derivation of an expression for resonant wavelength

The parameter used to express the shape of the cavity will be R, defined as L/D where L and D are the length and "average diameter" respectively of the cavity. The following definitions are made:

(33) k, the wave number, $k = 2\pi/\lambda$

(34) $k_1 = r\,e/q$

(35) $k^2 = k_1^2 + k_3^2$

(36) $p = \pi/2$

---

31 This s should not be confused with the s used in Tables of Mathieu Functions which is equal to c^2, nor with the s used in Kinzer and Wilson which is the reciprocal of the λ_c/s used here.

32 This E (e) should not be confused with the E used by Kinzer and Wilson to denote ellipticity.

33 B. O. Peirce, A Short Table of Integrals, Ginn, 1929, p 121

It follows from equations (35) and (22) that

(37) $$k_1^2 = (r/q)^2 - \omega^2\mu\epsilon$$

and for propagation, k_1 must be a pure imaginary and $\omega^2\mu\epsilon$ is greater than the quantity $(r/q)^2$,

(38) $$k_1 = i\,k_3 = i\sqrt{\omega^2\mu\epsilon - (r/q)^2}$$

(39) $$k_3^2 = \omega^2\mu\epsilon - (r/q)^2$$

At the resonant frequency, k_3 is zero so that

(40) $\omega_c^2\mu\epsilon = (r/q)^2$ and, after solving for λ_c

(41) $$\lambda_c = 2\pi q/r$$

For the TE_{111} mode,

(42) $$k_3 = \pi/L$$

When this value for k_3 is set equal to the value for k_3 from equation (39), and the equation is solved for λ, the resulting equation is:

(43) $$\lambda = \frac{1}{\sqrt{(1/2L)^2 + (r/2\pi q)^2}}$$

When the value from equation (41) is substituted into equation (43) and both sides of the equation are divided by D to make the resulting expression dimensionless, the equation obtained is:

(44) $$\lambda/D = \frac{\pi}{\sqrt{(pR)^2 + (1/\lambda_c/s)^2}}$$

The values of the parameter λ_c/s are plotted as a function of eccentricity in Chu.[35] It has already been noted that the values of λ_c/s obtained by Chu are not sufficiently accurate to be used in determining values for the quality factor, but the values for the resonant wavelength from equation (44) can be evaluated very easily, without the use of Mathieu function tables, if Chu's values for λ_c/s are used. Figure 4 is a plot of equation (44) using Chu's values. It is noted that these resonant wavelength values are not sufficiently accurate and do not enter directly into the values for the quality factor.[36]

If a more precise determination of the resonant wavelength is desired, it is necessary to combine equations (31) and (41) to obtain the relation which was used in evaluating the quality factor:

(45) $$(r)\,(\lambda_c/s) = (\pi/2)\,/\,E\,(e)$$

When the eccentricity is equal to zero, $E\,(e)$ is equal to $\pi/2$ so that λ_c/s becomes the reciprocal of the root.[37] If both sides of equation (44) are multiplied by D and the substitution is

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[35] loc cit

[36] It should be observed that the variation in resonant wavelength does affect Q since it appears in the k^3 term in the numerator of equation (82)

[37] Kinzer refers to his parameter "s" which is the reciprocal of $\lambda c/s$ as the "root value adjusted to the eccentricity."

made for zero eccentricity from equation (45), the equation obtained is:

(46)
$$\lambda_{circ} = \frac{2}{\sqrt{(r/pD)^2 + (1/L)^2}}$$

This agrees with the formula for wavelength for the circular resonant cavity given by Montgomery.[38]

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38 C. G. Montgomery, ed., Technique of Microwave Measurements, MIT Radiation Laboratory Series, McGraw-Hill, 1947, p 297

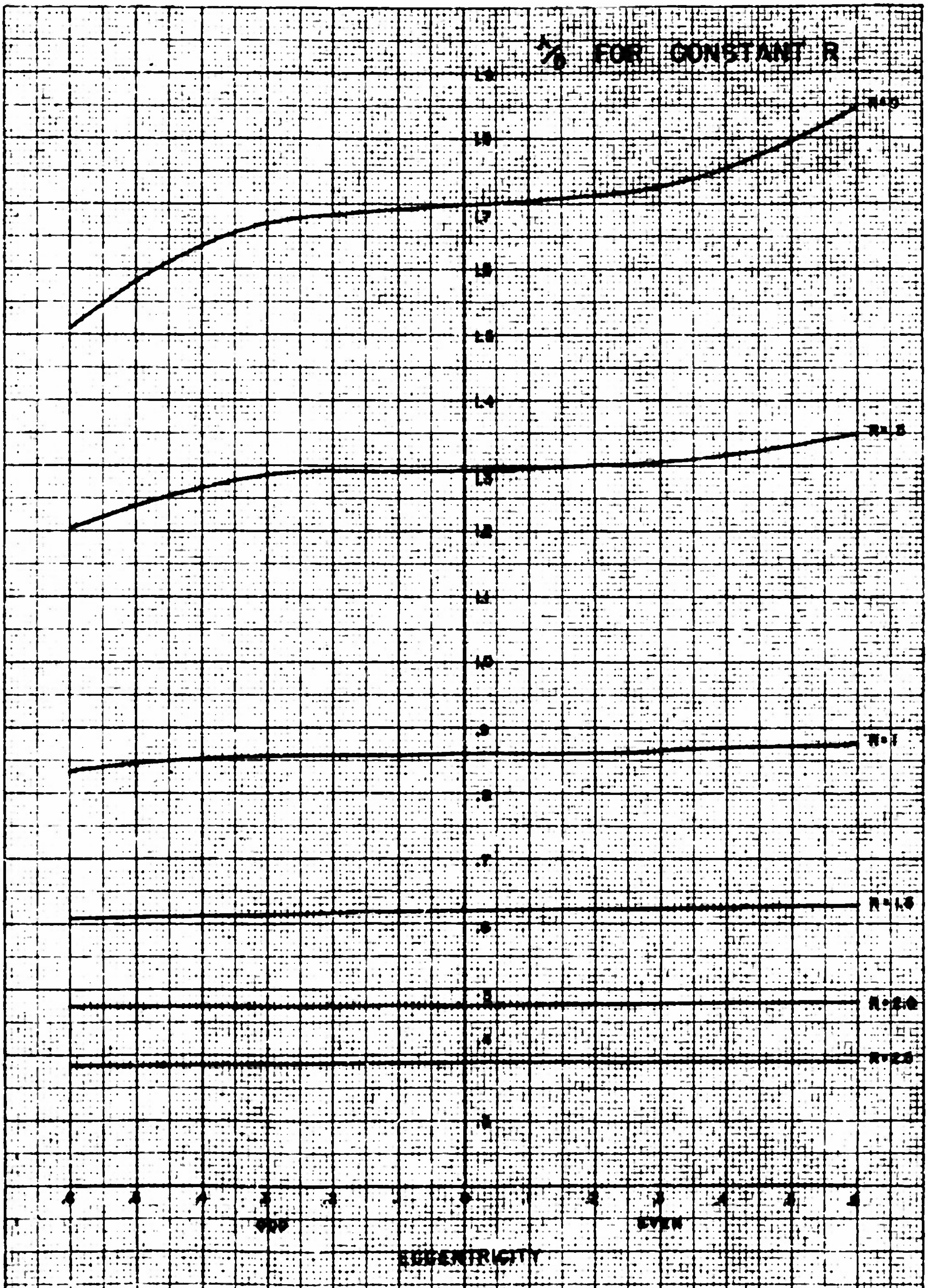

FIGURE 4

CHAPTER V

DETERMINATION OF THE QUALITY FACTOR, Q

(1) Derivation of an expression for the quality factor, Q

The quality factor, Q, of a resonant cavity is ordinarily defined as the ratio of the product of the angular frequency and the energy stored to the average power loss.[39] The total energy in the electric and magnetic fields remains constant (neglecting losses) and the maximum electric energy is equal to the maximum magnetic energy. It is sufficient to consider either the electric or magnetic stored energy. Since one component of the electric field is absent in the TE_{111} mode, it is more convenient to consider the stored electric energy. This can be found by integrating the E_ξ and E_η components of energy over this volume.

(47) $$Q = \frac{\omega U}{\text{Power Lost}}$$

The stored electric energy,

(48) $$U_\epsilon = \int_{\eta=0}^{\eta=2\pi} \int_{\xi=0}^{\xi=\xi_0} \int_{z=0}^{z=L} \frac{\epsilon}{2} \left[|E_\eta^2| + |E_\xi^2| \right] ds_3 ds_2 ds_1$$

The differential element in the z direction, ds_3, is equal to dz, and the only function of z which appears in the expressions for E_ξ and E_η [40] is $\sin(k_3 z)$ or $\sin(\pi z/L)$ so the integral

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[39] S. Ramo and J. R. Whinnery, Fields and Waves in Modern Radio, Wiley, 1944, p 378

[40] Equations (25) and (26)

with respect to z is:

(49) $$\int_0^L \sin^2 (\pi z/L)\, dz = L/2$$

This reduces the expression for stored electric energy to the double integral:

(50) $$U_\epsilon = \int_{\eta=0}^{\eta=2\pi} \int_{\xi=0}^{\xi=\xi_0} \frac{L\epsilon}{4} \left[|E_\eta{}^2| + |E_\xi{}^2| \right] ds_2 ds_1$$

Substituting from equations (8) and equations (26) and (27):

(51) $$U_\epsilon = \int_{\eta=0}^{\eta=2\pi} \int_{\xi=0}^{\xi=\xi_0} \frac{L\epsilon B^2 \mu k^2}{4\epsilon} \left(\left[Sp_1(\eta) \right]^2 \left[Jp_1'(\xi) \right]^2 \right) \frac{ds_2}{q_1} \frac{ds_1}{q_1}$$

$$+ \int_{\eta=0}^{\eta=2\pi} \int_{\xi=0}^{\xi=\xi_0} \frac{L\epsilon B^2 \mu k^2}{4\epsilon} \left(\left[Sp_1'(\eta) \right]^2 \left[Jp_1(\xi) \right]^2 \right) \frac{ds_2}{q_1} \frac{ds_1}{q_1}$$

The radial and angular functions are entire functions of ξ and η respectively, and each double integral may be written as the product of two single integrals. It is noted from equation (8) that ds_2 is equal to $q_1\, d\eta$ and ds_1 is equal to $q_1\, d\xi$. When these substitutions are made and the terms independent of the variables of integration are removed, the expression for the stored energy can be written:

(52) $$U_\epsilon = \frac{L\,B^2\mu k^2}{4} \int_{\eta=0}^{\eta=2\pi} \left[Sp_1(\eta)\right]^2 d\eta \int_{\xi=0}^{\xi=\xi_o} \left[Jp_1'(\xi)\right]^2 d\xi$$

$$+ \int_{\eta=0}^{\eta=2\pi} \left[Sp_1'(\eta)\right]^2 d\eta \int_{\xi=0}^{\xi=\xi_o} \left[Jp_1(\xi)\right]^2 d\xi$$

The following abbreviations for integrals will be adopted:

(53) $$Ip = \int_{\eta=0}^{\eta=2\pi} \left[Sp_1(\eta)\right]^2 d\eta$$

(54) $$Ip' = \int_{\eta=0}^{\eta=2\pi} \left[Sp_1'(\eta)\right]^2 d\eta$$

(55) $$IIp = \int_{\xi=0}^{\xi=\xi_o} \left[Jp_1(\xi)\right]^2 d\xi$$

(56) $$IIp' = \int_{\xi=0}^{\xi=\xi_o} \left[Jp_1'(\xi)\right]^2 d\xi$$

These integrals cannot be evaluated analytically and the evaluations must be made by a combination of integration of series and numerical methods. The actual evaluation of the integrals is discussed in Chapter VI.

The expression for the stored electric energy is written using these abbreviations as:

(57) $$U_\epsilon = \frac{L\,B^2\mu k^2}{4} \left[Ip\,IIp' + Ip'\,IIp\right]$$

The power loss in the resonant cylinder is due to the copper losses from the currents flowing in the side wall and in the end plates. The average power loss is:

(58) $$\int_{\text{surface}} \left(\frac{|H^2|}{2} R_s \right)$$

where R_s is the surface resistivity as defined in Ramo and Whinnery.[41] In the side wall the ξ component of current is zero, since, by equation (24),

(59) $$H_\xi(\xi_o) = 0$$

The other two components of current present in the side wall are evaluated with $\xi = \xi_o$ and when squared are:[42]

(60) $$H_z^2 = B^2 k_1^4 \left[Sp_1(\eta)\right]^2 \left[Jp1(\xi_o)\right]^2 \sin^2(k_3 z)$$

(61) $$H_\eta^2 = \frac{B^2 k_3^2}{q_1^2} \left[Sp_1(\eta)\right]^2 \left[Jp_1(\xi_o)\right]^2 \cos^2(k_3 z)$$

These values must be integrated over the side wall which requires integration from $z = 0$ to $z = L$ and from $\eta = 0$ to $\eta = 2\pi$. The value of the integral in equation (49) is the same for an integrand of either the sine or cosine function squared so that the integration over the z range of either of equations (60) or (61) yields $L/2$.

The integral of H_z over the side wall may now be written:

(62) $$\int_{\text{side wall}} |H_z|^2 = \frac{B^2 k_1^4 \left[Jp_1(\xi_o)\right]^2 L}{2} \int_0^{2\pi} \left[Sp_1(\eta)\right]^2 ds_2$$

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[41] loc. cit. p 209

[42] from equations (23) and (25)

The integrand is multiplied and divided by q_1, and the substitutions from equations (7) and (8) evaluated at $\xi = \xi_o$

(63) $$q_1 = q(\cosh^2\xi_o - \cos^2\eta)^{\frac{1}{2}}$$

(64) $$d\,\eta = ds_2 / q_1$$

are made so that (63) becomes:

(65) $$\int_{\text{sidewall}} |H_z|^2 = \frac{B^2 k_1^4 \left[Jp_1(\xi_o)\right]^2 L}{2} \int_0^{2\pi} q(\cosh^2\xi_o - \cos^2\eta)^{\frac{1}{2}}\left[Sp_1(\eta)\right]^2 d\eta$$

Factoring out $q \cosh\xi_o$ from the integrand and substituting $e = 1/\cosh\xi_o$ in the integrand, equation (65) becomes

(66) $$\int_{\text{sidewall}} |H_z|^2 = \frac{B^2 k_1^4 \left[Jp_1(\xi_o)\right]^2 L\, q \cosh\xi_o}{2} \int_0^{2\pi} (1 - e^2\cos^2\eta)^{\frac{1}{2}}\left[Sp_1(\eta)\right]^2 d\eta$$

A further abbreviation is made:[43]

(67) $$IIIp = \int_0^{2\pi} (1 - e^2\cos^2\eta)^{\frac{1}{2}}\left[Sp_1(\eta)\right]^2 d\eta$$

and then the integral of H_z over the side wall is written:

(68) $$\int_{\text{sidewall}} |H_z|^2 = \tfrac{1}{2} B^2 k_1^4 \left[Jp_1(\xi_o)\right]^2 L\, q \cosh\xi_o\, IIIp$$

The integral of the square of the η component of H over the side wall is written:

(69) $$\int_{\text{sidewall}} |H_\eta|^2 = \frac{B^2 k_3^2 \left[Jp_1(\xi_o)\right]^2 L}{2} \int_0^{2\pi} \frac{1}{q_1}\left[Sp_1(\eta)\right]^2 \frac{ds_2}{q_1}$$

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[43] Admittedly there is a plethora of integral abbreviation symbols. However, the final expression for Q would require three pages if written without these abbreviations, so they must be accepted as a necessity rather than a confusing convenience.

The substitutions from equations (63) and (64) are made to give:

(70) $$\int_{\text{sidewall}} |H_\eta|^2 = \frac{B^2 k_3^2 \left[Jp_1(\xi_o)\right]^2 L}{2} \int_0^{2\pi} \frac{1}{q(\cosh^2\xi_o - \cos^2\eta)^{\frac{1}{2}}} \left[Sp_1'(\eta)\right]^2 d\eta$$

which becomes

(71) $$\int_{\text{sidewall}} |H_\eta|^2 = \frac{B^2 k_3^2 \left[Jp\ (\xi_o)\right]^2 L}{2q \cosh\xi_o} \int_0^{2\pi} (1 - e^2\cos^2\eta)^{-\frac{1}{2}} \left[Sp_1'(\eta)\right]^2 d\eta$$

when $1/q \cosh \xi_o$ is factored out of the integrand and the substitution $e = 1/\cosh \xi_o$ is made in the integrand.

The further integral abbreviation is made:

(72) $$IVp' = \int_0^{2\pi} (1 - e^2\cos^2\eta)^{-\frac{1}{2}} \left[Sp_1'(\eta)\right]^2 d\eta$$

Then the integral of H_η can be written:

(73) $$\int_{\text{sidewall}} |H_\eta|^2 = \frac{B^2 k_3^2 \left[Jp_1(\xi_o)\right]^2 L}{2q \cosh \xi_o} \ IVp'$$

Substituting from equations (68) and (73) into the expression for average power loss, equation (58), the expression for average power loss in the side wall is obtained:

(74) $$PL_{sw} = R_s \frac{B^2 L}{4} \left[Jp_1(\xi_o)\right]^2 \left[k_1^4 \, q \cosh\xi_o \, IIIp + \frac{k_3^2}{q \cosh\xi_o} IVp'\right]$$

This may be put in a somewhat more convenient form:

(75) $$PL_{sw} = R_s \frac{B^2 L}{4} \frac{k_1^2}{q \cosh\xi_o} \left[Jp_1(\xi_o)\right]^2 \left[\frac{k_1^2}{(q \cosh\xi_o)^2} IIIp + \left(\frac{k_3}{k_1}\right)^2 IVp'\right]$$

It is seen from equation (23) that the z component of H is zero at the end walls where $z = 0$ or $z = \pi$. When equations (24) and (25) are evaluated at either of the end walls the square of the current in one wall is:

(76) $$|H_\xi|^2 = \frac{B^2 k_3^2}{q_1^2} \left[Sp_1(\eta)\right]^2 \left[Jp_1(\xi)\right]^2$$

$$|H_\eta|^2 = \frac{B^2 k_3^2}{q_1^2} \left[Sp_1(\eta)\right]^2 \left[Jp_1(\xi)\right]^2$$

When the substitutions $ds_1 = q_1 \, d\xi$ and $ds_2 = q_1 \, d\eta$ are made, the integral equations become:

(77) $$\int_{\text{endwall}} |H_\xi|^2 = B^2 k_3^2 \int_0^{\xi_0}\int_0^{2\pi} \left[Sp_1(\eta)\right]^2 \left[Jp_1'(\xi)\right]^2 d\eta \, d\xi$$

(78) $$\int_{\text{endwall}} |H_\eta|^2 = B^2 k_3^2 \int_0^{\xi_0}\int_0^{2\pi} \left[Sp_1'(\eta)\right]^2 \left[Jp_1(\xi)\right]^2 d\eta \, d\xi$$

The integral abbreviations stated in equations (53), (54), (55), and (56) are used. The fact that there are two end walls provides a two which cancels the factor of one-half in equation (58) so that the total power loss in the end walls becomes:

(79) $$PL_{ew} = B^2 k_3^2 R_s \left(IpIIp' + Ip'IIp\right)$$

Equations (57), (74), and (79) provide the information required to substitute in the equation:

$$Q = \frac{\omega\, U}{\text{Average Power Loss}}$$

The angular frequency, ω, can be expressed in terms of k by using the relation:

$$\omega = \frac{2\pi}{\lambda\sqrt{\mu\epsilon}} = \frac{k}{\sqrt{\mu\epsilon}}$$

from equation (33).

This gives the formula for Q:

(80) $$Q = \frac{\sqrt{\mu/\epsilon}}{R_s} \frac{\frac{L}{4} k^3 \left(I_pII_p' + I_p'II_p\right)}{\frac{L\, k_1^2}{4q\cosh\xi_o}\left[Jp_1(\xi_o)\right]^2 \left[\frac{(k_1)^2\, III_p}{(q\cosh\xi_o)^2} + \left(\frac{k_3}{k_1}\right)^2 IV_p'\right] + k_3^2\left(I_pII_p' + I_p'II_p\right)}$$

It will be convenient to consider $Q\,\delta/\lambda$ instead of Q.[44] The relation used will be:

(81) $$\text{if } Q = \frac{\sqrt{\mu/\epsilon}}{R_s} A, \text{ then } Q\,\delta/\lambda = A/\pi$$ [45]

Also the substitutions are made from equations (36), (42), and (34):

$$p = \pi/2$$

$$L/2 = p/k_3$$

$$a/e = r/k_1$$

[44] This will make direct comparison with published curves for Q for the circular case possible.

[45] Ramo and Whinnery, Fields and Waves in Modern Radio, Wiley, 1944, p 211

(82) $$Q\,\delta/\lambda = \frac{1}{2\pi}\;\frac{k^3\,(IpIIp' + Ip'IIp)}{\frac{k_1^3}{2r}\left[Jp_1(\xi_0)\right]^2\left[r^2 IIIp + \left(\frac{k_3}{k_1}\right)^2 IVp'\right] + \frac{k_3^3}{p}\,(IpIIp' + Ip'IIp)}$$

Another integral abbreviation is made:

(83) $$Vp = IpIIp' + Ip'IIp$$

From equations (33) and (44):

(84) $$k = 2\pi/\lambda = \frac{2}{D}\left[\left(p^2R^2 + \left(\frac{1}{\lambda_0/s}\right)^2\right)\right]^{\frac{1}{2}}$$

and from the equations on the preceding page and the relation $R = D/L$

(85) $$k_3^3 = \frac{8\,p^3R^3}{D^3}$$

and, finally, using equations (34) and (33)

(86) $$k_1 = r\,e/q = 4r\,E(e)/s = \frac{2}{D\,\lambda_0/s}$$

When the substitutions from equations (83), (84), (85), and (86) are made in equation (82), and the numerator and denominator are simplified, the equation for $Q\,\delta/\lambda$ used for making calculations is obtained.

$$Q\,\delta/\lambda = \frac{V_p}{\pi r\,\mathrm{III}p\left[J_{p_1}(r)\right]^2}\;\frac{\left(1 + (pR\lambda_o/s)^2\right)^{3/2}}{1 + \dfrac{(pR\lambda_o/s)^2\,\mathrm{IV}p}{r^2\,\mathrm{III}p} + \dfrac{2(\lambda c/s)^3\,\mathrm{V}p\,p^2R^3}{\left[J_p(r)\right]^2 r\,\mathrm{III}p}} \tag{87}$$

A demonstration that equation (87) reduces to the formula for the circular case at zero eccentricity will be postponed until after the evaluation of the integrals is discussed in Chapter VI.

CHAPTER VI

CALCULATION OF THE QUALITY FACTOR, Q

(1) Determination of the roots and quantities which follow directly.

The first step in the calculation of the quality factor, Q, is the determination of the roots of the boundary condition equation (22) for the values of c which are used. It develops that for small values of eccentricity (between zero and 0.5) that the value of the parameter c varies between zero and one in this mode. The new and more accurate Tables Relating to Mathieu Functions [46] cannot be used because the only values of coefficients falling in the desired range are for values of c of zero, 0.707, and 1.0.[47] The Elliptic and Spheroidal Wave Functions [48] provide the coefficients for values of c at intervals of 0.2 accurate to five significant figures.

Cambi's Eleven Place Tables of Bessel Functions [49] were used to evaluate the Bessel functions. The coefficients for the Mathieu functions are zero beyond D_5 for the range of c considered in both of the series (15) and (16) so that Bessel functions of the first, third, and fifth order were the only ones required. It is desirable to determine the root values to five significant figures, but the arguments for Bessel functions [50]

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46 loc. cit. pages 58 and 155
47 values of e of zero, 0.5, and 1. Refer to table on page 12
48 loc. cit.
49 E. Cambi, Eleven and Fifteen-Place Tables of Bessel Functions, Dover, 1948
50 The same interval is used in Jahnke and Emde, Tables of Functions, Dover, 1945

are given in Cambi to increments of 0.01 in the neighborhood of the roots. To determine the roots, auxilary tables were made using linear interpolation for Bessel functions of the first, third, and fifth orders in increments of 0.001 from 1.80 to 2.10. This gave values of the needed Bessel functions for three hundred values of the argument in the desired range.

The roots of the radial functions are given to an accuracy of 0.01 in Tang.[51] For the even mode, using the roots of Tang as a starting place, values of the radial function,

(15) $$Je_1(c \cosh\xi) = \sqrt{\pi/2}\sum_{k=0}^{\infty}(-1)^k De^1_{2k+1}\, \mathcal{J}_{2k+1}(c \cosh\xi)$$

were plotted against $c \cosh \xi$ and a maximum was found graphically. This maximum occurs at the root, r, (which is equal to $c \cosh \xi_o$). When the root is found for a given value of c, then $\cosh \xi_o$ and ξ_o can be determined, and the eccentricity is the reciprocal of $\cosh \xi_o$.

For the odd mode the roots must be determined somewhat differently since the term $\tanh \xi$ appears in the formula for the radial function,

(16) $$Jo_1(c \cosh\xi) = \sqrt{\pi/2}\,\tanh\xi \sum_{k=0}^{\infty}(-1)^k(2k)Do^1_{2k+1}\,\mathcal{J}_{2k+1}(c \cosh\xi)$$

This function was plotted against values of ξ and the maximum determined. The WPA <u>Tables of Circular and Hyperbolic Sines and Cosines</u>[52] was

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[51] Loc. cit. page 43
[52] <u>Tables of Circular and Hyperbolic Sines and Cosines</u>, Work Projects Administration, New York, 1940

used to evaluate F. Steps of the argument are 0.0001 so that no interpolation was required. When F_c is determined, the eccentricity and the root, r, can be determined by a process which is the reverse of that used for the even mode.

Kinser and Wilson [53] give root values for the even mode to five significant figures. They make no comment on the probable error in these figures; this would lead to the assumption that a more accurate interpolation formula was used than a linear one. The even roots obtained by this author by the method outlined above agree with the Kinser and Wilson values within one ten-thousandth. Actual calculations were made using the Kinser and Wilson values. Kinser and Wilson did not determine values for the odd mode of the TE_{111} mode. According to Scarborough [54] these root values obtained using linear interpolation should be considered as subject to a possible error of two ten-thousandths. It will be seen later [55] that the error present in the root value dominates all the other errors present in the values for Q.

The values of the complete elliptic integral from equation (13) are tabulated in Peirce [56] with arc sin e as the argument. Linear interpolation was used to find the value of E (e) with a possible error of 0.00005.

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53 loc. cit. page 428
54 Scarborough, Numerical Analysis, McGraw-Hill
55 Chapter VII
56 loc. cit. page 121

Next the parameter λ_0/s was determined using the formula derived from equation (45)

(88) $$\lambda_0/s = \frac{\pi/2}{r\,E\,(e)}.$$

All of the roots and the quantities which are found directly from the roots are tabulated at the end of this chapter.

(2) Evaluation of the integrals

i) Evaluation of $Ip = \int_0^{2\pi} \left[Sp(\eta)\right]^2 d\eta$

From equation (13):

$$Se_1(c, \cos\eta) = \sum_{k=0}^{\infty} De^1_{2k+1} \cos\left[(2k+1)\eta\right]$$

This expression can be squared and integrated term by term. Because of the orthogonality of the trigonometric functions, cross products will give zero over the full range and,

(89) $$\int_0^{2\pi} \cos^2 mx\, dx = \int_0^{2\pi} \sin^2 mx\, dx = \pi$$ [57]

so that the value of the integral will be the sum of the squares of the coefficients. That is

(90) $$Ie = \pi\left[(De^1_1)^2 - (De^1_3)^2 - (De^1_5)^2 - \cdot \cdot \cdot\right]$$

The De_r are tabulated in Stratton, Morse, Chu, and Hutner.[58]

57 B. O. Peirce, A Short Table of Integrals, formula 489
58 loc. cit. page 78

It should be noted that for calculations in which it is possible to use the Tables Relating to Mathieu Functions, this value can be found directly from the tables tabulated as $N_r = \pi/A^2$.

The integral for the odd function differs only in the trigonometric function involved and the numerical value of the coefficients. The Do_r are tabulated in Stratton [60] on page 82.

$$I_o = \pi\left[(Do_1^1)^2 + (Do_3^1)^2 + (Do_5^1)^2 + \cdots\right] \tag{91}$$

These integrals are also evaluated directly in the Tables Relating to Mathieu Functions for values of c as $N_r' = \pi/B^2$.

ii) Evaluation of $I_p' = \int_0^{2\pi}\left[Sp_1'(\eta)\right]^2 d\eta$

The expression in equation (13) can be differentiated term by term to give:

$$Se_1'(c\ \cos\eta) = \sum_{k=0}^{\infty} -De_{2k+1}^1\ (2k+1)\ \sin\left[(2k+1)\eta\right] \tag{92}$$

for the even mode, and for the odd mode:

$$So_1'(c\ \cos\eta) = \sum_{k=0}^{\infty} Do_{2k+1}^1\ (2k+1)\ \cos\left[(2k+1)\eta\right] \tag{93}$$

The De_r in equation (92) and the Do_r in equation (93) are the same coefficients as those in equations (90) and (91) respectively. The values of the integrals are therefore:

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60 loc. cit. as F_1. See table on page 12

(94) $$Ie' = \pi\left[(De_1^1)^2 + (3De_3^1)^2 + (5De_5^1)^2 + \cdots\right]$$

(95) $$Io' = \pi\left[(Do_1^1)^2 + (3Do_3^1)^2 + (5Do_5^1)^2 + \cdots\right]$$

iii) Evaluation of IIp $= \int_0^{\xi_0} \left[Jp_1\,(c\cosh\xi)\right]^2 d\xi$

From equation (15)

(15) $$Je_1(c\cosh\xi) = \sqrt{\pi/2}\sum_{k=0}^{\infty}(-1)^k\, De_{2k+1}\, \mathcal{J}_{2k+1}(c\cosh\xi)$$

For a given c, the interval to the corresponding ξ_0 was divided into 24 parts, and values for $Je_1(c\cosh\xi)$ were calculated for each of these points. Each of the values was squared and these values were integrated numerically using Simpson's one-third rule:[61]

$$\text{INTEGRAL} = \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + \cdots + y_{23}) + 2(y_2 + y_4 + \cdots + y_{22}) + y_{24}\right]$$

where h is the difference between successive abscissas (that is $\xi_0/24$ in this particular case) and the y's are the ordinates. The combination of the error in the coefficients and the Bessel functions was calculated to be 0.00005. The inherent error in Simpson's formula for integration is given as:

(96) $$E_S = \frac{(\xi_0 - 0)}{180}\left(\frac{\xi_0}{24}\right)^4 f^{iv}(\xi)$$

where f^{iv} is the fourth derivative of the function and may be

61 L. Scarborough, Numerical Analysis, page 176

approximated by the fourth differences. The fourth differences are of the order of 0.00002 so that the inherent error is clearly negligible with the coefficients used.

Kinzer and Wilson[62] used Weddle's formula for integration. The inherent error is considerably less than the error with Simpson's formula when the integrated function has sixth derivatives that are a great deal less than the fourth differences. Since the value of the coefficients are accurate to only five significant figures, the additional difficulty in using Weddle's rule seems unjustified.

The estimated error in these integrals is twice the sum of the product of the root and the error in determination of the functions, and the product of the average of the function value and the error in finding the root. This error was calculated as 0.0003.

The integral IIo is calculated in exactly the same manner as IIe, using equation (16)

$$Jo_1(c \cosh \xi) = \sqrt{\pi/2}\ \tanh\xi \sum_{k=0}^{\infty} (-1)^k (2k) Do^1_{2k+1}\, \mathcal{J}_{2k+1}(c \cosh \xi) \tag{16}$$

iv) Evaluation of $IIp' = \int_0^{\xi_0} \left[Jp_1'(\xi) \right]^2 d\xi$

The expressions (15) and (16) are differentiated and evaluated at each of the 25 points which were found in iii). The functions are squared and the same integral formula is used to perform the integration. The error calculated was 0.0005 which is larger than the calculated error for the integrals IIp because of the error in

62 loc. cit. page 430

finding the derivative.

v) Evaluation of IIIp = $\int_0^{2\pi}(1 - e^2\cos^2\eta)^{\frac{1}{2}}\left[Sp(\eta)\right]^2 d\eta$

The term $(1 - e^2\cos^2\eta)^{\frac{1}{2}}$ may be expanded by a binomial expansion to give

(98) $$\left[1 - (1/2)e^2\cos^2\eta - (1/8)e^4\cos^4 - \cdot \cdot \cdot \right]$$

All but the first two terms of the expansion are neglected and substituting from equation (13)

(99) $$IIIe = \int_0^{2\pi}(1 - \tfrac{1}{2}e^2\cos^2\eta)\left[Se(\eta)\right]^2 d\eta$$

If the trigonometric substitution

$$\cos^2\eta = \tfrac{1}{2}(1 + \cos 2\eta)$$

is made, then

$$IIIe = \int_0^{2\pi}\left[1 - \tfrac{1}{4}{}^2(1 + \cos 2\eta)\right]\left[Se(\eta)\right]^2 d\eta$$

(100) $$IIIe = (1 - \tfrac{1}{4}e^2)\int_0^{2\pi}\left[Se(\eta)\right]^2 d\eta - \tfrac{1}{4}e^2\int_0^{2\pi}\left[Se(\eta)\right]^2 \cos 2\eta \, d\eta$$

The integral expressed by equation (100) is evaluated in McLachlan.[63]

(101) $$IIIe = (1 - \tfrac{1}{4}e^2)\pi - \frac{\pi}{4}{}^2\left[\tfrac{1}{2}(De_1^1)^2 + \sum_{k=0}^{\infty}(De_{2k+1}^1)(De_{2k+3}^1)\right]$$

63 loc. cit. page 79

A relation for the odd mode is obtained in the same manner as for the even mode:

(102) $$\mathrm{IIIo} = (1-\tfrac{1}{4}e^2)\int_0^{2\pi}\left[So(\eta)\right]^2 d\eta - \tfrac{1}{4}e^2\int_0^{2\pi}\left[So(\eta)\right]^2 \cos\, 2\eta\, d\eta$$

The integral (102) is also evaluated by McLachlan [64]

(103) $$\mathrm{IIIo} = (1-\tfrac{1}{4}e^2)\pi - \frac{\pi}{4}e^2\left[-\tfrac{1}{2}(Do_1^1)^2 + \sum_{k=0}^{\infty}(Do^1_{2k+1})(Do^1_{2k+3})\right.$$

vi) Evaluation of $\mathrm{IV}p' = \int_0^{2\pi}(1-e^2\cos^2\eta)^{-\frac{1}{2}}\left[Sp'(\eta)\right]^2 d\eta$

As in v), a binomial expansion is used which in this case gives:

(104) $$1 + \tfrac{1}{2}e^2\cos^2\eta + \tfrac{3}{8}e^4\cos^2\eta - \cdots$$

Neglecting all but the first two terms of the binomial expansion,

(105) $$\mathrm{IV}e' \doteq \int_0^{2\pi}(1+\tfrac{1}{4}e^2)\left[Se'(\eta)\right]^2 d\eta + \int_0^{2\pi}\tfrac{1}{4}e^2\cos 2\eta\left[Se'(\eta)\right]^2 d\eta$$

which is evaluated by McLachlan as

(106) $$\mathrm{IV}e' = (1+\tfrac{1}{4}e^2)\pi + \pi\tfrac{1}{4}e^2\left[-\tfrac{1}{2}(De_1^1)^2 + \sum_{k=0}^{\infty}De^1_{2k+1}De^1_{2k+3}(2k+1)(2k+3)\right.$$

Similarly for the odd case,

(107) $$\mathrm{IV}o' = (1+\tfrac{1}{4}e^2)\pi + \pi\tfrac{1}{4}e^2\left[\tfrac{1}{2}(Do_1^1)^2 + \sum_{k=0}^{\infty}Do^1_{2k+1}Do^1_{2k+3}(2k+1)(2k+3)\right.$$

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64 loc. cit. page 79

(3) The degenerate ellipse

When the eccentricity of the ellipse is zero, the constant c is also zero. All of the coefficients are zero except the first one so that the angular functions reduce to trigonometric functions and the radial functions reduce to first order Bessel functions. Reference to equations (90), (91), (94), and (95) shows that the integrals Ie, Io, Ie', and Io' all will have the value π at zero eccentricity. Further, consideration of equations (99), (102), (105), and (107) shows that IIIe, IIIo, IVe', and IVo' have the value at zero eccentricity.

Substituting from equation (83)

$$Vp = \pi(IIp' + IIp)$$

and (IIp' + IIp) is, in the degenerate case:

(108) $$\int_0^{\xi_0}\left[\mathcal{J}p_1(\xi)\right]^2 + \left[\mathcal{J}p_1(\xi)\right]^2 \, d\xi$$

Equation (108) is a special form of a Lommel integral and may be evaluated directly to give:[65]

(109) $$Vp = \frac{\pi}{2} \; r\left[J_1(r)\right]^2\left[1-(1/r)^2\right]$$

The values for the integrals are substituted into equation (87) and the substitution for the circular case from equation (45) that

$$r = 1/\lambda_0/s$$

[65] N. W. McLachlan, Bessel Functions for Engineers, Oxford, 1944, page 94

is made to get:

$$(110)\qquad Q\,\frac{\delta}{\lambda}\,\mathrm{circ} = \frac{1}{2\pi}\;\frac{(p^2R^2 + r^2)^{3/2}\,(1 - 1/r^2)}{r^2 + \dfrac{p^2R^2}{r^2} - \dfrac{p^2R^3}{r^2} + p^2R^3}$$

$$(111)\qquad Q\,\frac{\delta}{\lambda}\,\mathrm{circ} = \frac{1}{2\pi}\;\frac{(p^2R^2 + r^2)^{3/2}\,(1 - 1/r^2)}{r^2 + p^2R^3 + (1 - R)\,(pR/r)^2}$$

The equation (111) agrees with the formula given by Montgomery for the circular cavity in the TE_{111} mode.[66]

[66] C. G. Montgomery ed., Technique of Microwave Measurements, MIT Radiation Laboratory Series, McGraw Hill, 1947, p 300, eqn 35

TABLE I

The Root Values and The Associated Quantities for the Even Mode

c	0	0.2	0.4	0.6	0.8	1.0
r	1.8412	1.8416	1.8430	1.8452	1.8484	1.8527
θ	0.0	0.10860	0.21704	0.32516	0.43280	0.53975
λ_0/s	0.543124	0.544494	0.549164	0.557018	0.568021	0.585972
III_e	3.14159	3.12758	3.08614	3.01715	2.92922	2.74550
IV_e'	3.14159	3.14621	3.15934	3.17923	3.20025	3.23751
I_e	3.14159	3.14976	3.17320	3.21404	3.27436	3.35585
I_e'	3.14159	3.14979	3.17383	3.21737	3.28523	3.38372
II_e	0.44689	0.45821	0.48853	0.51456	0.54168	0.47966
II_e'	0.19443	0.17233	0.13689	0.10291	0.06451	0.03499
$[J_e'(r)]^2$	0.53102	0.53331	0.53819	0.54616	0.55780	0.56722
coef Q	0.20456	0.205813	0.206384	0.207941	0.209873	0.192042
coef R^2	0.214702	0.216978	0.224271	0.236928	0.254519	0.291053
coef R^3	9.64278	9.64996	9.616895	9.55305	9.48998	9.06385

TABLE II

The Root Values and The Associated Quantities for the TE_{111} Mode

c	0	0.2	0.4	0.6	0.8	1.0	1.2
r	1.8412	1.8751	1.9082	1.9560	2.0075	2.0751	2.1651
ρ	0.0	0.1063	0.2113	0.3064	0.4000	0.4809	0.5531
λ_c/s	0.54312	0.53477	0.53006	0.52399	0.52001	0.51355	0.49497
IIIo	3.14159	3.13723	3.12482	3.10826	3.08954	3.07606	3.07210
IVo'	3.14159	3.15485	3.19403	3.25186	3.32977	3.41290	3.49823
Io	3.14159	3.16550	3.23791	3.36122	3.54324	3.79099	4.11913
Io'	3.14159	3.16552	3.23854	3.36455	3.55414	3.81876	4.17998
IIo	0.44689	0.43561	0.39614	0.36458	0.31817	0.27810	0.24357
IIo'	0.19443	0.21650	0.23594	0.29106	0.32391	0.36353	0.41524
$[J_0(r)]^2$	0.53102	0.52874	0.52351	0.51378	0.49973	0.48019	0.45900
coef Q	0.20456	0.203011	0.208682	0.218826	0.234021	0.253380	0.284492
coef R^2	0.214702	0.201814	0.294606	0.185251	0.178433	0.167672	0.153956
coef R^3	9.64278	0.430000	0.481811	0.501074	0.510163	0.532039	0.534851

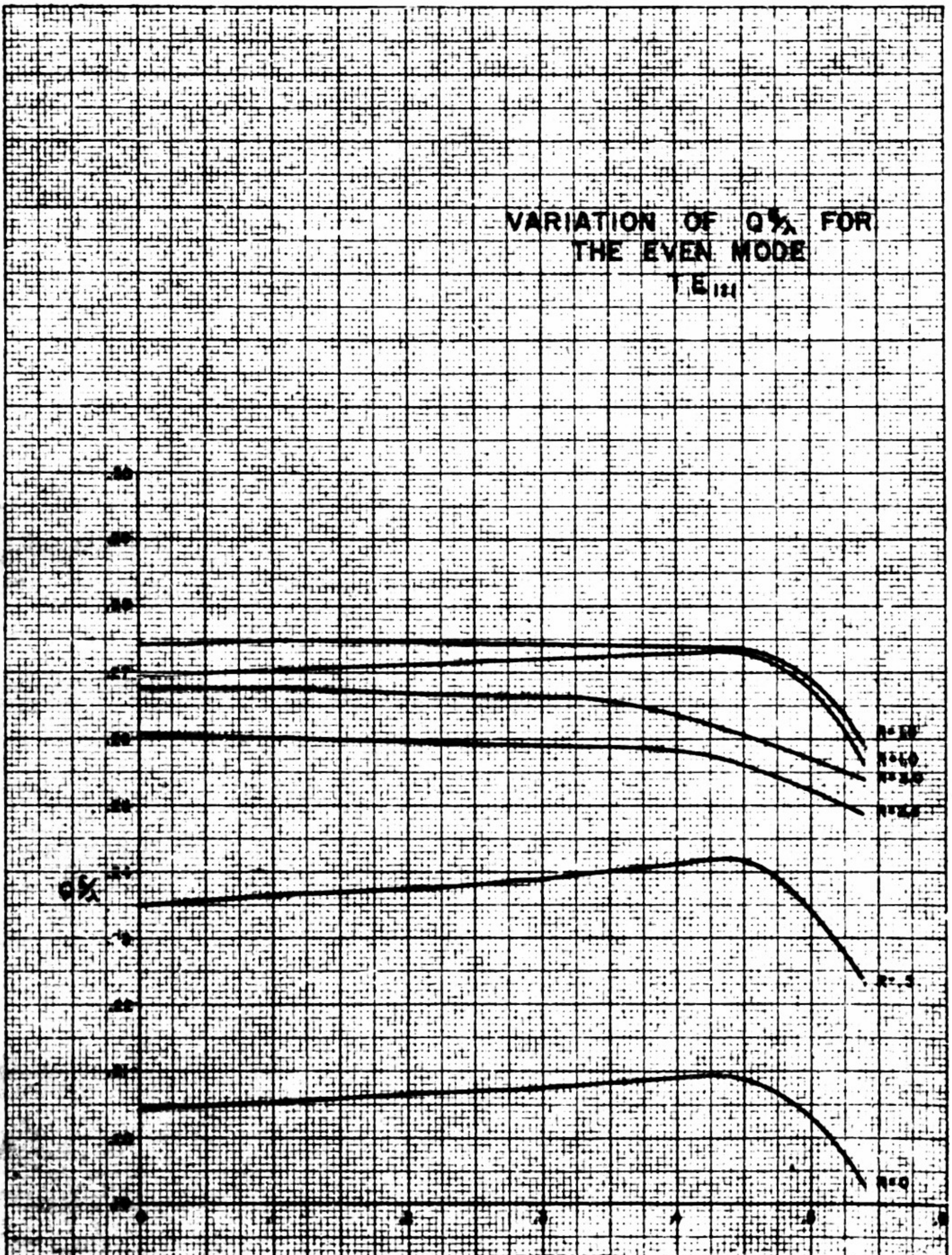
VARIATION OF Q δ/λ FOR
THE EVEN MODE
TE 111
Q δ/λ
ECCENTRICITY
R=.5
R=0

FIGURE 6

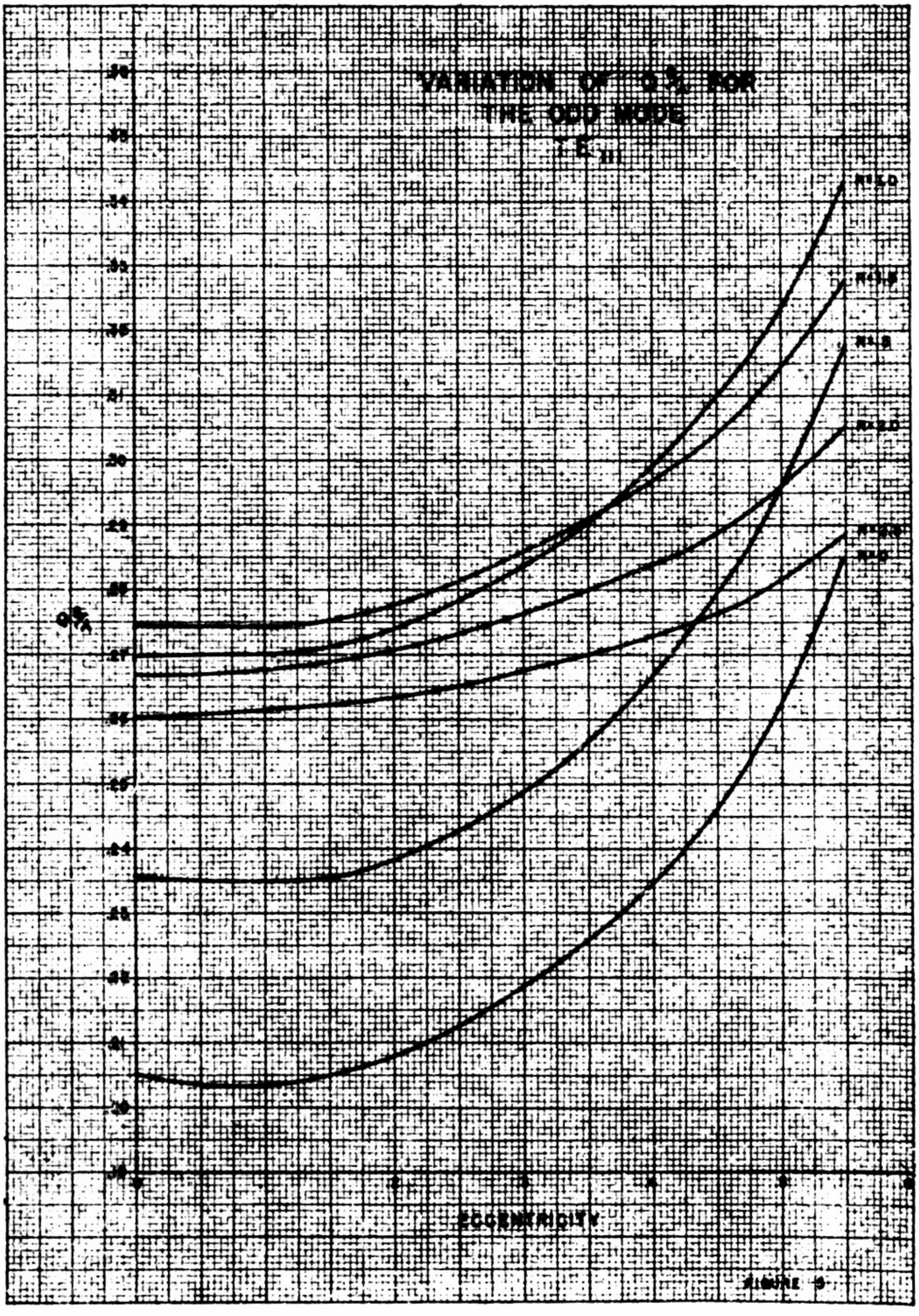
VARIATION OF Q% FOR
THE ODD MODE
TE_{11}
ECCENTRICITY

TABLE III

Table of Values of the Quality Factor, $Q\,\delta/\lambda$, for the TE_{111} Modes

Even Mode

R \ c	0.0	0.2	0.4	0.6	0.8	1.0
0.0	0.20456	0.20581	0.20638	0.20794	0.20987	0.19204
0.5	0.23529	0.23670	0.23752	0.23954	0.24210	0.22322
1.0	0.26968	0.27074	0.27101	0.27191	0.25509	0.25509
1.5	0.27402	0.27453	0.27411	0.27365	0.25879	0.25879
2.0	0.26714	0.26736	0.26673	0.26591	0.25382	0.25382
2.5	0.26009	0.26010	0.25958	0.25863	0.24905	0.24905

Odd Mode

R \ c	0.0	0.2	0.4	0.6	0.8	1.0	1.2
0.0	0.20456	0.203011	0.206682	0.218826	0.234021	0.253380	0.284482
0.5	0.23529	0.234981	0.239178	0.249231	0.266107	0.286589	0.317875
1.0	0.26968	0.269917	0.274278	0.284091	0.298339	0.316165	0.342376
1.5	0.27402	0.274031	0.278940	0.286029	0.296394	0.308909	0.327308
2.0	0.26714	0.267602	0.271507	0.276831	0.283785	0.292366	0.304984
2.5	0.26009	0.261693	0.263727	0.267844	0.272745	0.278940	0.287949

CHAPTER VII

INTERPRETATION OF THE CURVES FOR THE QUALITY FACTOR

(1) Analysis of the error in the calculations.

The curves for the quality factor are plotted in Figures 5 and 6. The extremely small changes in the quality factor, Q, at small values of eccentricity make a careful consideration of the accuracy range necessary. The limits of accuracy in the determination of the roots and evaluation of the integrals was noted when these evaluations were discussed in Chapter VI. It is now desirable to see how these component errors are reflected in the final result.

The error calculations are simple and intermediate steps are omitted. All quotients are changed to product form by expressing the denominator with a negative exponent. All quantities to fractional exponents are put into the form $(a + e)^{m/n}$ and then expanded by a binomial expansion to two terms. All numbers are rounded to two significant figures.

The error in the determination of the derivative root value was assumed to be 0.00005. This may also be considered as the error appearing in λ_0/s. It develops that this error is the dominating one. It is therefore designated by the symbol e' while all the other errors are denoted by multiples of e where e also signifies an error of 0.00005.

To determine the error in k^3:

$$k^3 \approx \left[1 + (\lambda_\sigma/s + e')^2 p^2 R^2\right]^{3/2}$$

$$\approx 2.4R^3(1 + 1.3e') \qquad ^{67}$$

The error in evaluation of the integrals I and II was assumed to be 0.00005, and the error in the evaluation of the integrals I' and II' was considered as 0.0005. The first of these errors is designated by e and the second by 10 e. The error in the evaluation of V can be written as:

$$V = I\,II' + I'II$$

$$\approx (I + e)(II + 10e) + (I' + 10e)(II + e)$$

$$\approx I\,II' + I'\,II + 66e$$

The error in the calculation of the integral III was assumed to be 0.00005 so that:

$$V/III \approx \frac{2 + 66e}{3 + e}$$

$$\approx 2/3 + 28e$$

The error in the coefficient of Q, which is independent of R, can then be written:

$$\text{coef } Q \approx \frac{V}{3\,r\,III\,\,J(r)^2}$$

$$\approx \frac{2/3 + 28e}{3\,(2 + e')(.5 + e)}$$

$$\approx 2/9 + .1e' + 9.4\,e$$

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67 The equations for estimating error are not a developmental step and for this reason they are not numbered

Similarly it was found that for the R^2 coefficient:

$$\text{coef } R^2 \approx 5 + 5e' + 3e$$

and for the R^3 coefficient:

$$\text{coef } R^3 \approx .5 + 17e + 13e'$$

For $R = 1$

$$Q\,\delta/\lambda \approx .3 + 15.1e' + 19e$$

$$\approx .3 - .0017 \qquad \text{(for both } e' \text{ and } e \text{ taken as } 0.00005)$$

For $R = 2.5$

$$Q\,\delta/\lambda \approx .3 + 6e' + 6.3\,e$$

$$\approx .3 + .0007 \qquad \text{(for both } e' \text{ and } e \text{ taken as } 0.00005)$$

It is noted that the error in the determination of the root introduces about the same error in the final result as the sum of all the other errors in the determinations of the integrals. If a more conservative estimate of the error in the determination of the root is made[68] of, say, 0.0002, the estimated error in the determination of $Q\,\delta/\lambda$ becomes 0.004.

(2) <u>The auxiliary component curves</u>

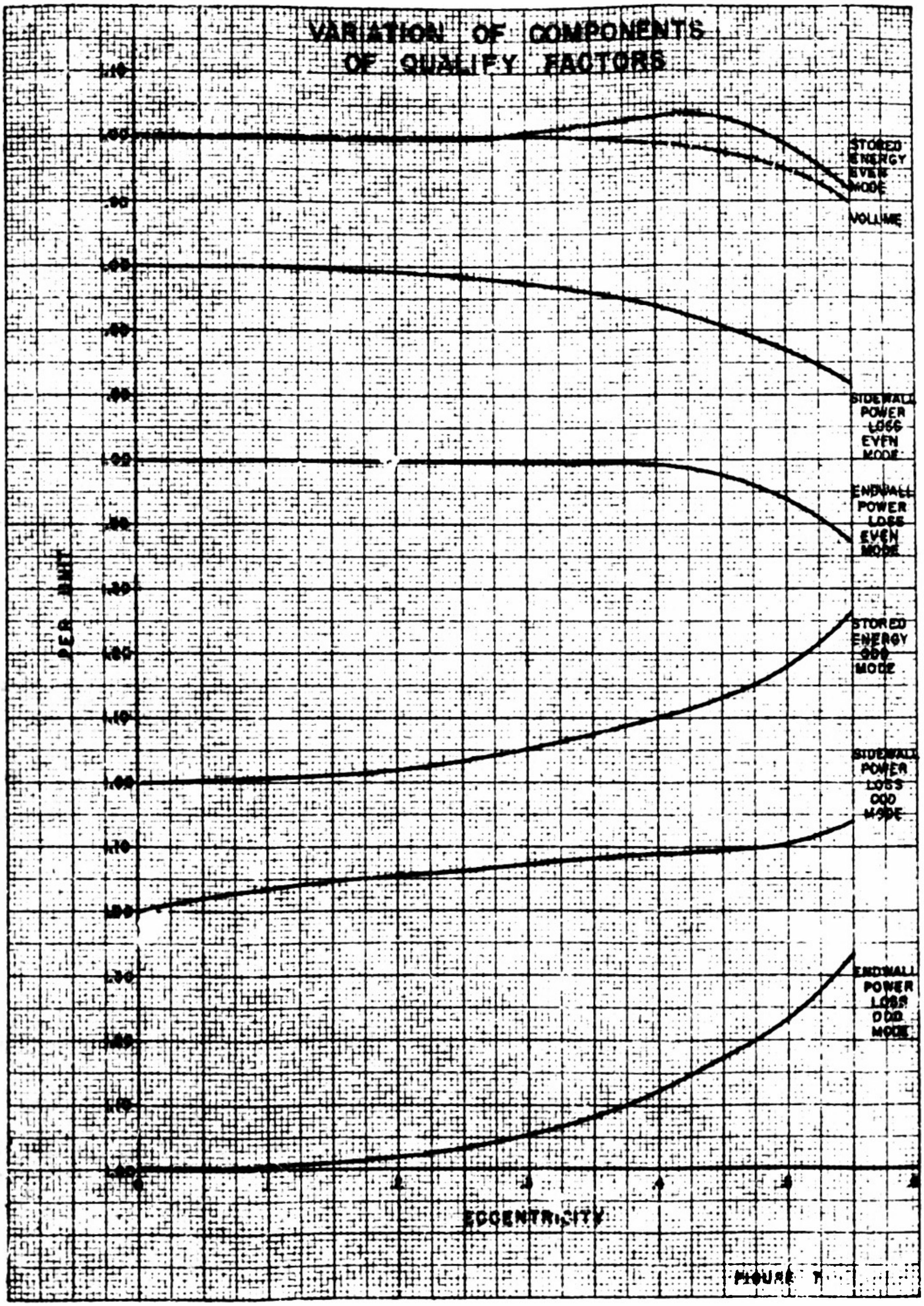
When a problem is arranged for maximum facility in numerical computation, the significant components may be mixed so that it

[68] See page 34

becomes difficult to see the relations of the individual changes that combine to cause the overall result. Inspection of the curves for Q in figures 5 and 6 shows that the quality factor varies very slightly for values of eccentricity less than 0.4, but a cursory examination of the data in Tables I, II, and III shows that these slight changes have not been the result of small changes in the components, but, rather, of compensating changes of much larger magnitudes than the resulting change in the quality factor, Q.

To examine them qualitatively, the individual variations in the stored energy, the power loss in the end walls, and the power loss in the sidewall are plotted in Figure 7 to slide rule accuracy. The characteristics depend on the ratio of the average diameter to the length so that it is necessary to choose a fixed value of R; the convenient value of R = 1 is chosen. The curves in Figure 7 are based on unit magnitude at zero eccentricity, and they provide no information concerning the relative magnitude of the changes which occur.

The volume of the cavity is directly proportional to the product of the major and minor axes. For a given perimeter, the cross section area is a maximum at zero eccentricity, and it may seem odd that the stored energy increases when the volume is decreased. At small values of eccentricity, the volume decreases very little as the circle is deformed; a plot of the volume on a per unit basis is



VARIATION OF COMPONENTS OF QUALIFY FACTORS

FIGURE 7

included in Figure 7 for comparison with the change in the stored energy. The increase in stored energy can be attributed to the change in resonant wavelength, and, perhaps, a slightly more efficient distribution of the fields in the cavity with the change in eccentricity. Although this change in distribution would be difficult to demonstrate mathematically, it seems to have some intuitive justification. The Mathieu function at small eccentricities approximates a slightly distorted Bessel function. The lack of symmetry of the dominating first order Bessel function in the range from zero to the first derivative root makes it seem possible that a slightly non-symmetric cylinder might use such a function more efficiently than the perfect cylinder could.

At an eccentricity of 0.5 in the odd mode, it is noted that the stored energy is still increasing. Certainly this increase could not continue indefinitely and must change to a decrease when the volume begins to decrease rapidly.

(3) Discussion of Results and Conclusions

The numerical values of $Q\,\delta/\lambda$ for zero eccentricity agree numerically with those plotted by Montgomery for the circular cylindrical cavity.[69]

The plot of the quality factor in Figure 5 for the odd mode shows that the quality factor may decrease in value slightly for very small

[69] Montgomery, E. G. ed., Technique of Microwave Measurements, MIT Radiation Laboratory Series, 1947, page 301

amounts of eccentricity. This decrease is of the same order as the possible error in calculations and should not be given undue consideration. It seems reasonable to conclude that the quality factor remains constant for values of eccentricity less than 0.25 for excitation in either the even or odd mode.

Ordinarily, a deformed circular cylinder will be excited at the same time in both the odd and even modes. If the excitation orientation can be controlled relative to the deformation, it will be preferable to excite the even mode, since both the wavelength and quality factor change less in that mode.

It is unfortunate that other modes could not be evaluated by an approximate means; most of the individual quantities involved in the expression for the quality factor were calculated by Tang [70] but calculations made using his values have a possible error range that is larger than the magnitude of the change in Q. Calculations would be simplified a little for modes for which the newer Tables Relating to Mathieu Functions could be used since some of the integrals are evaluated directly in those tables.

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[70] loc. cit.

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